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ON THE STABILITY OF A NONAUTONOMOUS HAMILTONIAN SYSTEM WITH TWO DEGREES OF FREEDOM

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Stability of the equilibrium position of a nonautonomous Hamiltonian system with two degrees of freedom is investigatied for the resonant case. The conditions of instability as well as those of formal stability are obtained.

1. We assume that the coordinate origin $q_i = p_i = 0$ corresponds to the position of equilibrium of the canonical system of differential equations

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i} \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i} \qquad (i = 1,2) \tag{1.1}$$

where H is a Hamiltonian function 2π-periodic in t and analytic in the vicinity of the point $q_i = p_i = 0$.

Let the linearized system be stable and all its multipliers be distinct. We assume that the Hamiltonian in (1.1) is transformed into

$$H = {}^1/_2\, \lambda_1 (q_1^2 + p_1^2) + {}^1/_2\, \lambda_2 (q_2^2 + p_2^2) + \sum_{\nu=3}^{\infty} h_{\nu_1\nu_2\nu_3\nu_4}(t)\, q_1^{\nu_1} q_2^{\nu_2} p_1^{\nu_3} p_2^{\nu_4} \tag{1.2}$$

by means of a real linear 2π-periodic canonical transformation [1]. In (1.2) $\pm i\lambda_1$ and $\pm i\lambda_2$ are the characteristic indices of the linearized system and ν_i are nonnegative integers

$$\nu = \nu_1 + \nu_2 + \nu_3 + \nu_4, \qquad h_{\nu_1\nu_2\nu_3\nu_4}(t + 2\pi) = h_{\nu_1\nu_2\nu_3\nu_4}(t)$$

We also assume that the condition

$$k_1\lambda_1 + k_2\lambda_2 \not\equiv 0 \qquad (\text{mod } 1) \tag{1.3}$$

holds for the integers k_1 and k_2 satisfying the equalities $|k_1| + |k_2| = 3$ or $|k_1| + |k_2| = 4$. Then there exists [2] an analytic canonical transformation 2π-periodic in t, reducing the Hamiltonian (1.2) to the form

$$H = \lambda_1 r_1 + \lambda_2 r_2 + l_{2020} r_1^2 + l_{1111} r_1 r_2 + l_{0202} r_2^2 + O(|q|^5) \tag{1.4}$$

$$\left(|q| = \sqrt{q_1^2 + q_2^2 + p_1^2 + p_2^2}, \quad 2r_i = q_i^2 + p_i^2\right)$$

Coefficients $l_{\nu_1\nu_2\nu_3\nu_4}$ in (1.4) are independent of t. Let the quadratic form

$$l_{2020} r_1^2 + l_{1111} r_1 r_2 + l_{0202} r_2^2$$

be sign definite in the quadrant $r_1 \geqslant 0$, $r_2 \geqslant 0$. Then the position of equilibrium is formally stable [1, 3 and 4]. This means that for the system (1.1) there exists a power series, possibly divergent,

$$G = G_n(q_i, p_i, t) + G_{n+1}(q_i, p_i, t) + \dots \tag{1.5}$$

which formally will be a positive definite integral 2π-periodic in t. In other words, all coefficients of the power series

$$G_{q_1} H_{p_1} - G_{p_1} H_{q_1} + G_{q_2} H_{p_2} - G_{p_2} H_{q_2} + G_t \tag{1.6}$$

are identically equal to zero and

$$G_n(q_i, p_i, t) \geqslant 0$$

At the same time

$$G_n(q_i, p_i, t) = 0$$

only for $q_i = p_i = 0$.

Formal stability implies that retention of terms of arbitrarily high power in ν in the expansion (1.2) does not lead to detection of the Liapunov instability. Even if trajectories beginning at the coordinate origin exist, the progress along these trajectories is extremely slow [3 and 5–8].

In this paper we investigate the stability when condition (1.3) does not hold for $k_i \geqslant 0$. We assume that condition (1.3) does not hold for a single pair of nonnegative integers k_1 and k_2 satisfying the condition $k_1 + k_2 = 3$ or $k_1 + k_2 = 4$, i.e. we investigate nonmultiple resonances.

Thus we shall consider the following nine resonant cases:

$$\begin{array}{llll} (1)\ 3\lambda_1 = m, & (2)\ 3\lambda_2 = m, & (3)\ \lambda_1 + 2\lambda_2 = m \\ (4)\ 2\lambda_1 + \lambda_2 = m, & (5)\ 4\lambda_1 = m, & (6)\ 4\lambda_2 = m \\ (7)\ 2(\lambda_1 + \lambda_2) = m, & (8)\ \lambda_1 + 3\lambda_2 = m, & (9)\ 3\lambda_1 + \lambda_2 = m \end{array} \tag{1.7}$$

where m is an integer.

Since the multipliers are assumed distinct, the integral and half-integral values of λ_i as well as those satisfying the equation $\lambda_1 \pm \lambda_2 \equiv 0 \pmod 1$ are not considered. This means that the stability of (1.1) is studied within the region of stability of the linearized system.

2. We shall first consider the stability in the cases (1)–(4). Applying a real, analytic transformation 2π-periodic in t (see Sect. 4) we can eliminate all third degree terms in the Hamiltonian (1.2) other than resonant. In the new variables q_i^* and p_i^* the Hamiltonian becomes

$$H^* = H_2^* + H_3^* + O(|q|^4)$$

Here $H_2^* = \frac{1}{2}\lambda_1 (q_1^{*2} + p_1^{*2}) + \frac{1}{2}\lambda_2 (q_2^{*2} + p_2^{*2})$, and expressions H_3^* for the cases (1)–(4) of (1.7) are

$$\begin{aligned}
&(1)\quad H_3^* = 2u_{0030}^* (q_1^{*3} - 3q_1^* p_1^{*2}) - 2v_{0030}^* (p_1^{*3} - 3p_1^* q_1^{*2}) \\
&(2)\quad H_3^* = 2u_{0003}^* (q_2^{*3} - 3q_2^* p_2^{*2}) - 2v_{0003}^* (p_2^{*3} - 3p_2^* q_2^{*2}) \\
&(3)\quad H_3^* = -2u_{0012}^* [q_1^* (p_2^{*2} - q_2^{*2}) + 2p_1^* q_2^* p_2^*] - 2v_{0012}^* [p_1^* (p_2^{*2} - q_2^{*2}) - 2q_1^* q_2^* p_2^*] \\
&(4)\quad H_3^* = -2u_{0021}^* [q_2^* (p_1^{*2} - q_1^{*2}) + 2p_2^* q_1^* p_1^*] - 2v_{0021}^* [p_2^* (p_1^{*2} - q_1^{*2}) - 2q_2^* q_1^* p_1^*]
\end{aligned} \tag{2.1}$$

respectively.

The following notation is introduced in (2.1):

$$u_{\nu_1\nu_2\nu_3\nu_4}^* = x_{\nu_1\nu_2\nu_3\nu_4} \cos mt + y_{\nu_1\nu_2\nu_3\nu_4} \sin mt, \qquad v_{\nu_1\nu_2\nu_3\nu_4}^* = y_{\nu_1\nu_2\nu_3\nu_4} \cos mt - x_{\nu_1\nu_2\nu_3\nu_4} \sin mt$$

$$x_{\nu_1\nu_2\nu_3\nu_4} = \frac{1}{2\pi} \int_0^{2\pi} (u'_{\nu_1\nu_2\nu_3\nu_4} \cos mt - v'_{\nu_1\nu_2\nu_3\nu_4} \sin mt)\, dt \tag{2.2}$$

$$y_{\nu_1\nu_2\nu_3\nu_4} = \frac{1}{2\pi}\int_0^{2\pi} (u'_{\nu_1\nu_2\nu_3\nu_4}\sin mt + v'_{\nu_1\nu_2\nu_3\nu_4}\cos mt)\,dt$$

Expressions for $u'_{\nu_1\nu_2\nu_3\nu_4}$ and $v'_{\nu_1\nu_2\nu_3\nu_4}$ are given in Sect. 4.

For each of the resonant cases (1)–(4) the following theorem holds.

Theorem 2.1. If $x^2_{\nu_1\nu_2\nu_3\nu_4} + y^2_{\nu_1\nu_2\nu_3\nu_4} \neq 0$, the position of equilibrium is unstable.

We shall prove this for case (1). After the canonical transformation

$$q_i^* = p_i^\circ \sin(\lambda_i t - \theta) + q_i^\circ \cos(\lambda_i t - \theta)$$
$$p_i^* = p_i^\circ \cos(\lambda_i t - \theta) - q_i^\circ \sin(\lambda_i t - \theta) \qquad (i = 1, 2) \tag{2.3}$$

where

$$\sin 3\theta = \frac{y_{0030}}{\sqrt{x^2_{0030} + y^2_{0030}}}, \qquad \cos 3\theta = \frac{x_{0030}}{\sqrt{x^2_{0030} + y^2_{0030}}}$$

we obtain the Hamiltonian in the following form which is independent of t up to the third order terms inclusive:

$$H^\circ = 2\sqrt{x_{0030}^2 + y_{0030}^2}\quad (q_1^{\circ 3} - 3q_1^\circ p_1^{\circ 2}) + O(|q|^4)$$

Changing to polar coordinates

$$q_i^\circ = \sqrt{2r_i}\sin\varphi_i, \qquad p_i^\circ = \sqrt{2r_i}\cos\varphi_i \qquad (i = 1, 2) \tag{2.4}$$

we obtain

$$H^\circ = -4\sqrt{2(x^2_{0030} + y^2_{0030})}\, r_1 \sqrt{r_1}\sin 3\varphi_1 + O(|q|^4) \tag{2.5}$$

Using now Chetaev theorem [9] to prove the instability, we take the function V in the form $V = V_1 V_2$, where

$$V_1 = r_1^\alpha - r_2^2, \; V_2 = r_1\sqrt{r_1}\cos 6\varphi_1 \quad (\alpha > 2) \tag{2.6}$$

and define the region $V > 0$ by $(V_1 > 0,\ -\pi/12 < \varphi_1 < \pi/12)$. At the boundary of this region either V_1 or V_2 is equal to zero and within the region the following equality holds:

$$r_2 = \beta r_1^{\alpha/2} \qquad (0 < \beta < 1) \tag{2.7}$$

We choose the parameter α so, that the derivative of V (by virtue of the equations of motion containing the Hamiltonian (2.5)), is positive definite in the region $V > 0$.

It can easily be verified that for $2 < \alpha < 3$

$$\frac{dV}{dt} = 6\sqrt{2(x^2_{0030} + y^2_{0030})}\, r_1^{\alpha+2}\{[2\alpha\cos 3\varphi_1 + f_1]\cos 6\varphi_1 + 3(1 - \beta^2)\times$$
$$\times[\cos 3\varphi_1 + \sin 3\varphi_1 \sin 6\varphi_1 + f_2]\} \tag{2.8}$$

where f_1 and f_2 become arbitrarily small when r_1 tends to zero.

In the region $V > 0$ we have $\cos 3\varphi_1 > \sqrt{2}/2$ and $\cos 3\varphi_1 + \sin 3\varphi_1 \sin 6\varphi_1 \geqslant 1$.

It therefore follows from (2.7) and (2.8) that for sufficiently small $|q|$ the function dV/dt is positive definite in the region $V > 0$, and by the Chetaev theorem the position of equilibrium is unstable.

In case (3), applying (2.3) and (2.4) where we now have

$$\sin 3\theta = \frac{y_{0012}}{\sqrt{x^2_{0012} + y^2_{0012}}}, \qquad \cos 3\theta = \frac{x_{0012}}{\sqrt{x^2_{0012} + y^2_{0012}}}$$

we obtain

$$H^\circ = -4\sqrt{2(x^2_{0012} + y^2_{0012})}\, r_2\sqrt{r_1}\sin(\varphi_1 + 2\varphi_2) + O(|q|^4) \tag{2.9}$$

In order to prove the instability, the Chetaev function should be taken in the form $V = V_1 V_2$, where

$$V_1 = r_2^\alpha - (r_2 - 2r_1)^2, \; V_2 = r_2\sqrt{r_1}\cos 2(\varphi_1 + 2\varphi_2) \quad (2 < \alpha < 3) \tag{2.10}$$

and the region $V > 0$ should be defined by $(V_1 > 0,\ -\pi/4 < \varphi_1 + 2\varphi_2 < \pi/4)$.

Proof of the theorem for the cases (2) and (4) is analogous to that of (1) and (3), respectively.

3. Next we shall consider the stability in the cases (5)−(9). Here the transformed Hamiltonian in polar coordinates has the form

$$H^{\circ} = l_{2020}r_1^2 + l_{1111}r_1r_2 + l_{0202}r_2^2 - H^{**}(r_i,\ \varphi_i) + H'(r_i,\ \varphi_i,\ t) \tag{3.1}$$

In (3.1) $H' = O(|q|^5)$ and the function H^{**} for the cases (5)−(9) is, respectively,

$$\begin{aligned}
&(5)\ H^{**} = \sqrt{x_{0040}^2 + y_{0040}^2}\, r_1^2 \sin 4\varphi_1 \\
&(6)\ H^{**} = \sqrt{x_{0004}^2 + y_{0004}^2}\, r_2^2 \sin 4\varphi_2 \\
&(7)\ H^{**} = \sqrt{x_{2200}^2 + y_{2200}^2}\, r_1 r_2 \sin 2(\varphi_1 + \varphi_2) \\
&(8)\ H^{**} = \sqrt{x_{1300}^2 + y_{1300}^2}\, r_2 \sqrt{r_1 r_2} \sin(\varphi_1 + 3\varphi_2) \\
&(9)\ H^{**} = \sqrt{x_{3100}^2 + y_{3100}^2}\, r_1 \sqrt{r_1 r_2} \sin(3\varphi_1 + \varphi_2)
\end{aligned} \tag{3.2}$$

When bringing the Hamiltonian to the form (3.1) we assume, that

$$x^2_{\nu_1\nu_2\nu_3\nu_4} + y^2_{\nu_1\nu_2\nu_3\nu_4} \neq 0$$

and in the formulas (2.3) we have

$$\sin 4\theta = -\frac{x_{\nu_1\nu_2\nu_3\nu_4}}{\sqrt{x^2_{\nu_1\nu_2\nu_3\nu_4} + y^2_{\nu_1\nu_2\nu_3\nu_4}}}, \qquad \cos 4\theta = -\frac{y_{\nu_1\nu_2\nu_3\nu_4}}{\sqrt{x^2_{\nu_1\nu_2\nu_3\nu_4} + y^2_{\nu_1\nu_2\nu_3\nu_4}}}$$

Formulas for $l_{\nu_1\nu_2\nu_3\nu_4}$, $x_{\nu_1\nu_2\nu_3\nu_4}$ and $y_{\nu_1\nu_2\nu_3\nu_4}$ are given in Sect. 4.

We introduce the quantities A_j and B_j ($j = 5,6,7,8,9$) for each of the resonant cases (5)−(9), defining them as follows:

$$\begin{aligned}
A_5 &= \sqrt{x_{0040}^2 + y_{0040}^2}, & B_5 &= l_{2020} \\
A_6 &= \sqrt{x_{0004}^2 + y_{0004}^2}, & B_6 &= l_{0202} \\
A_7 &= \sqrt{x_{2200}^2 + y_{2200}^2}, & B_7 &= l_{2020} + l_{1111} + l_{0202} \\
A_8 &= 3\sqrt{3(x_{1300}^2 + y_{1300}^2)}, & B_8 &= l_{2020} + 3l_{1111} + 9l_{0202} \\
A_9 &= 3\sqrt{3(x_{3100}^2 + y_{3100}^2)}, & B_9 &= 9l_{2020} + 3l_{1111} + l_{0202}
\end{aligned} \tag{3.3}$$

Theorem 3.1. If the inequalities $A_j \neq 0$ and $A_j > |B_j|$ hold simultaneously, the position of equilibrium is unstable. If $A_j < |B_j|$ and the Hamiltonian includes terms of up to the fourth order, the equilibrium is stable. If the function $H^{\circ} - H'$ is sign definite, the position of equilibrium is formally stable.

Let us prove the theorem for case (5) To prove the first statement of the theorem we take the Chetaev function in the form $V = V_1V_2$, where

$$V_1 = r_1^{\alpha} - r_2^2,\quad V_2 = r_1^{a}\cos 4a\varphi_1 \quad (a = 1 + \varepsilon,\ 2 < \alpha < 3,\quad 0 < \varepsilon \ll 1) \tag{3.4}$$

and define the region $V > 0$ by $(V_1 > 0, -\pi/8a < \varphi_1 < \pi/8a)$.

Within $V > 0$ we have

$$r_2 = \beta r_1^{\alpha/2} \qquad (0 < \beta < 1)$$

and the derivative is

$$\frac{dV}{dt} = 4r_1^{\alpha+3}\{(\alpha A_5 \cos 4\varphi_1 + g_1)\cos 4a\varphi_1 + 2(1-\beta^2)[A_5 \cos 4\varepsilon\varphi_1 - B_5 \sin 4a\varphi_1 + \varepsilon \sin 4a\varphi_1 (A_5 \sin 4\varphi_1 - B_5) + g_2]\} \tag{3.5}$$

where the functions g_1 and g_2 are arbitrarily small when r_1 tends to zero.

Since by definition we have $A_5 > |B_5|$, therefore at sufficiently small ε the function

dV/dt will be positive definite in the region $V > 0$ for sufficiently small $|q|$. This proves the statement on instability.

Second statement of the theorem is proved by constructing a Liapunov function for the truncated system with a Hamiltonian $H^\circ - H'$, possessing two integrals $r_2 =$ const and $H^\circ - H' =$ const. This function is taken in the form

$$W = r_2^4 + (H^\circ - H')^2 \tag{3.6}$$

It is easy to verify that the latter is positive definite for $A_5 < |B_5|$, therefore the position of equilibrium is stable [10].

To prove the last statement of the theorem we apply the transformations described in Sects. 2 and 4 to the Hamiltonian (1.2) reducing it formally to a function independent of t in all orders. Then the expression $G \equiv H^\circ$ will formally be the integral of (1.1), provided that it is written out in the initial variables q_i and p_i. We obtain

$$G = G_4 + G_5 + \ldots \qquad (G_4 \equiv H^\circ - H')$$

Therefore, if $H^\circ - H'$ is a sign definite function, the position of equilibrium is formally stable.

In case (7) we prove the instability with the aid of Chetaev function $V = V_1V_2$, where

$$V_1 = r_2^\alpha - (r_1 - r_2)^2, \qquad V_2 = r_1r_2 \cos 2a(\varphi_1 + \varphi_2) \qquad (a = 1 + \varepsilon,\ 2 < \alpha < 3,\ 0 < \varepsilon \ll 1) \tag{3.7}$$

defining the region $V > 0$ by $(V_1 > 0, -\pi/4a < \varphi_1 + \varphi_2 < \pi/4a)$. The stability in case (7) is proved with the aid of the Liapunov function

$$W = (r_1 - r_2)^4 + (H^0 - H')^2 \tag{3.8}$$

In case (8) we can take the function V in the form $V = V_1V_2$, where

$$V_1 = r_2^\alpha - (r_2 - 3r_1)^2, V_2 = r_2\sqrt{r_1r_2}\cos a(\varphi_1 + 3\varphi_2)\ (a = 1 + \varepsilon, 2 < \alpha < 3, 0 < \varepsilon \ll 1)$$

and define the region $V > 0$ by $(V_1 > 0, -\pi/2a < \varphi_1 + 3\varphi_2 < \pi/2a)$. Function W in this case can be taken in the form

$$W = (r_2 - 3r_1)^4 + (H^\circ - H')^2$$

Consideration of the resonances (6) and (9) is analogous to that of (5) and (8), respectively.

Notes. a) Resonances $k_1\lambda_1 + k_2\lambda_2 \equiv 0 \pmod 1$ for which $|k_1| + |k_2| \geqslant 5$ are not essential for the proof of Theorem 3.1 on formal stability.

b) If the equality $x^2_{\nu_1\nu_2\nu_3\nu_4} + y^2_{\nu_1\nu_2\nu_3\nu_4} = 0$ holds in cases (1)–(9), the presence of a resonance does not obstruct in reduction of the Hamiltonian to the form (1.4), and the criterion of formal stability given in Sect. 1 is applicable.

4. Here we give the computational formulas. Let the Hamiltonian in (1.1) have the form (1.2). We shall introduce new canonical variables q_i^* and p_i^* by means of the following generating function:

$$S = q_1p_1^* + q_2p_2^* + \sum_{\nu=3} s_{\nu_1\nu_2\nu_3\nu_4}(t)\, q_1^{\nu_1}q_2^{\nu_2}p_1^{*\nu_3}p_2^{*\nu_4}$$

Here $s_{\nu_1\nu_2\nu_3\nu_4}(t + 2\pi) = s_{\nu_1\nu_2\nu_3\nu_4}(t)$. Let us denote the new Hamiltonian by $H^*(q_i^*, p_i^*, t)$. We have the following identity:

$$H^*\left(\frac{\partial S}{\partial p_i^*}, p_i^*, t\right) \equiv H\left(q_i, \frac{\partial S}{\partial q_i}, t\right) + \frac{\partial S}{\partial t} \tag{4.1}$$

which yields

$$H_2^* = H_2, \quad H_3^* = H_3 + DS_3 \tag{4.2}$$

$$H_4^* = H_4 + \sum_{i=1}^{2} \frac{1}{2}\lambda_i\left[\left(\frac{\partial S_3}{\partial q_i}\right)^2 - \left(\frac{\partial S_3}{\partial p_i^*}\right)^2\right] + \frac{\partial H_3}{\partial p_i^*}\frac{\partial S_3}{\partial q_i} - \frac{\partial H_3^*}{\partial q_i}\frac{\partial S_3}{\partial p_i^*}$$

Here H_k, H_k^* and S_k are homogeneous kth degree functions in expansions into the power series of H, H^* and S, and D denotes the operator

$$D = \lambda_1\left(p_1^*\frac{\partial}{\partial q_1} - q_1\frac{\partial}{\partial p_1^*}\right) + \lambda_2\left(p_2^*\frac{\partial}{\partial q_2} - q_2\frac{\partial}{\partial p_2^*}\right) + \frac{\partial}{\partial t}$$

Let us choose the coefficients $s_{\nu_1\nu_2\nu_3\nu_4}$ such that the function H_3^* would assume its simplest form. Proceeding to complex conjugate canonic variables

$$q_k' = q_k + ip_k^*, \quad p_k' = q_k - ip_k^* \quad (k = 1, 2)$$

it is easy to show [2] that H_3^* can be eliminated if

$$a_{\nu_1\nu_2\nu_3\nu_4} = \lambda_1(\nu_3 - \nu_1) + \lambda_2(\nu_4 - \nu_2)$$

is not an integer. Simple manipulation yields the following expressions for the coefficients $s_{\nu_1\nu_2\nu_3\nu_4}$:

$$\begin{gathered}
s_{0300} = u''_{0003} + u''_{0102}, \quad s_{0102} = u''_{0102} - 3u''_{0003}, \quad s_{0201} = v''_{0102} + 3v''_{0003}, \\
s_{0003} = v''_{0102} - v''_{0003}, \quad s_{3000} = u''_{0030} + u''_{1020}, \quad s_{1020} = u''_{1020} - 3u''_{0030}, \\
s_{2010} = v''_{1020} + 3v''_{0030}, \quad s_{0030} = v''_{1020} - v''_{0030}, \quad s_{1002} = u''_{0111} - u''_{0012} - u''_{0210} \\
s_{1200} = u''_{0012} + u''_{0210} + u''_{0111}, \quad s_{0210} = v''_{0111} + v''_{0012} + v''_{0210}, \quad s_{0111} = 2(u''_{0210} - u''_{0012}) \\
s_{0012} = v''_{0111} - v''_{0012} - v''_{0210}, \quad s_{1101} = 2(v''_{0012} - v''_{0210}) \\
s_{0120} = u''_{1011} - u''_{0021} - u''_{2001}, \quad s_{2100} = u''_{0021} + u''_{2001} + u''_{1011} \\
s_{2001} = v''_{1011} + v''_{0021} + v''_{2001}, \quad s_{0021} = v''_{1011} - v''_{0021} - v''_{2001} \\
s_{1011} = 2(u''_{2001} - u''_{0021}), \quad s_{1110} = 2(v''_{0021} - v''_{2001})
\end{gathered} \tag{4.3}$$

$$\begin{gathered}
u''_{\nu_1\nu_2\nu_3\nu_4} = g(t)\sin a_{\nu_1\nu_2\nu_3\nu_4}t + f(t)\cos a_{\nu_1\nu_2\nu_3\nu_4}t \\
v''_{\nu_1\nu_2\nu_3\nu_4} = g(t)\cos a_{\nu_1\nu_2\nu_3\nu_4}t - f(t)\sin a_{\nu_1\nu_2\nu_3\nu_4}t
\end{gathered} \tag{4.4}$$

$$\begin{gathered}
g(t) = \operatorname{ctg}\pi a_{\nu_1\nu_2\nu_3\nu_4} I_1(2\pi) + I_2(2\pi) - 2I_2(t) \\
f(t) = I_1(2\pi) - \operatorname{ctg}\pi a_{\nu_1\nu_2\nu_3\nu_4} I_2(2\pi) - 2I_1(t)
\end{gathered} \tag{4.5}$$

$$\begin{gathered}
I_1(t) = \int_0^t (u'_{\nu_1\nu_2\nu_3\nu_4}\cos a_{\nu_1\nu_2\nu_3\nu_4}x - v'_{\nu_1\nu_2\nu_3\nu_4}\sin a_{\nu_1\nu_2\nu_3\nu_4}x)\,dx \\
I_2(t) = \int_0^t (u'_{\nu_1\nu_2\nu_3\nu_4}\sin a_{\nu_1\nu_2\nu_3\nu_4}x + v'_{\nu_1\nu_2\nu_3\nu_4}\cos a_{\nu_1\nu_2\nu_3\nu_4}x)\,dx
\end{gathered} \tag{4.6}$$

$$\begin{gathered}
u'_{0003} = {}^1/_8(h_{0300} - h_{0102}), \quad v'_{0003} = {}^1/_8(h_{0201} - h_{0003}) \\
u'_{0102} = {}^1/_8(h_{0102} + 3h_{0300}), \quad v'_{0102} = {}^1/_8(h_{0201} + 3h_{0003}) \\
u'_{0030} = {}^1/_8(h_{3000} - h_{1020}), \quad v'_{0030} = {}^1/_8(h_{2010} - h_{0030}) \\
u'_{1020} = {}^1/_8(h_{1020} + 3h_{3000}), \quad v'_{1020} = {}^1/_8(h_{2010} + 3h_{0030}) \\
u'_{0111} = {}^1/_4(h_{1200} + h_{1002}), \quad v'_{0111} = {}^1/_4(h_{0012} + h_{0210}) \\
u'_{0012} = {}^1/_8(h_{1200} - h_{1002} - h_{0111}), \quad v'_{0012} = {}^1/_8(h_{0210} - h_{0012} + h_{1101}) \\
u'_{0210} = {}^1/_8(h_{1200} - h_{1002} + h_{0111}), \quad v'_{0210} = {}^1/_8(h_{0210} - h_{0012} - h_{1101}) \\
u'_{1011} = {}^1/_4(h_{2100} + h_{0120}), \quad v'_{1011} = {}^1/_4(h_{2001} + h_{0021}) \\
u'_{0021} = {}^1/_8(h_{2100} - h_{0120} - h_{1011}), \quad v'_{0021} = {}^1/_8(h_{2001} - h_{0021} + h_{1110}) \\
u'_{2001} = {}^1/_8(h_{2100} - h_{0120} + h_{1011}), \quad v'_{2001} = {}^1/_8(h_{2001} - h_{0021} - h_{1110})
\end{gathered} \tag{4.7}$$

But if $a_{\nu_1\nu_2\nu_3\nu_4} = m$ (m is an integer), then the function H_3^* cannot be made to vanish identically. It can however be reduced to a normal form, reflecting the resonant character of the problem. In Sect. 2, the transformed function is given for particular cases of the resonance $a_{\nu_1\nu_2\nu_3\nu_4} = m$.

Function H_4^* can be simplified in a similar manner. Coefficients of the normal form required in the investigation of stability in the resonant cases (5)–(9) are

$$l_{2020} = \frac{1}{4\pi} \int_0^{2\pi} (h_{2020}^* + 3h_{0040}^* + 3h_{4000}^*)\, dt$$
$$l_{1111} = \frac{1}{2\pi} \int_0^{2\pi} (h_{2200}^* + h_{0220}^* + h_{2002}^* + h_{0022}^*)\, dt \tag{4.8}$$
$$l_{0202} = \frac{1}{4\pi} \int_0^{2\pi} (h_{0202}^* + 3h_{0004}^* + 3h_{0400}^*)\, dt$$

Coefficients $x_{\nu_1\nu_2\nu_3\nu_4}$ and $y_{\nu_1\nu_2\nu_3\nu_4}$ are obtained from (2.2), where we must set

$$\begin{gathered} u'_{0040} = 1/2\,(h_{0040}^* + h_{4000}^* - h_{2020}^*), \quad v'_{0040} = 1/2\,(h_{3010}^* - h_{1030}^*) \\ u'_{0004} = 1/2\,(h_{0004}^* + h_{0400}^* - h_{0202}^*), \quad v'_{0004} = 1/2\,(h_{0301}^* - h_{0103}^*) \\ u'_{1300} = 1/2\,(h_{1300}^* + h_{0013}^* - h_{1102}^* - h_{0211}^*), \quad v'_{1300} = 1/2\,(h_{0112}^* + \\ + h_{1003}^* - h_{0310}^* - h_{1201}^*), \quad u'_{3100} = 1/2\,(h_{3100}^* + h_{0031}^* - h_{1120}^* - h_{2011}^*) \\ v'_{3100} = 1/2\,(h_{1021}^* + h_{0130}^* - h_{2110}^* - h_{3001}^*), \quad u'_{2200} = 1/2\,(h_{0022}^* + \\ + h_{2200}^* - h_{0220}^* - h_{2002}^* - h_{1111}^*), \quad v'_{2200} = 1/2\,(h_{0121}^* + h_{1012}^* - h_{1210}^* - h_{2101}^*) \end{gathered} \tag{4.9}$$

Here $h^*_{\nu_1\nu_2\nu_3\nu_4}$ are the coefficients accompanying the corresponding powers in H_4^* computed according to the formula (4.2).

In conclusion we note certain inaccuracies which appeared in [11] in the proof of instability. Derivatives of the functions (2.8) and (3.8) of this paper can assume negative values near the boundaries of the corresponding regions $V > 0$ and, consequently, need not be positive definite in these regions. For $\omega_1 = 2\omega_2$ and $\omega_1 = 3\omega_2$, the value of V used in the present paper in the cases (3) and (8), respectively, should be adopted, and for $\omega_1 = 3\omega_2$ in the condition of instability the sign $\geqslant$ should be replaced by $>$.

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CONCERNING SOME SPACECRAFT CONVERGENCE CONTROL LAWS

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The problem of motion of an interceptor spacecraft along a three-dimensional trajectory in a central gravitational field is considered; this trajectory is the mapping in the involute plane of the shape, dimensions, and orientation of the Keplerian orbit of the target spacecraft. Control laws which yield analytical solutions of the encounter problem are chosen. The active spacecraft is referred to as the "interceptor", the passive spacecraft as the "target".

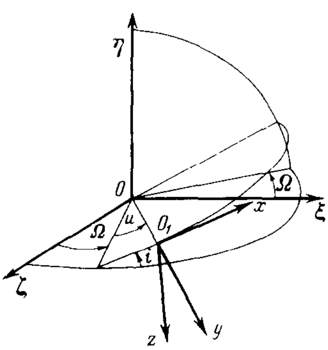

Fig. 1

1. The motion of the interceptor under the controlling acceleration W applied to its center of mass O_1 is described by equations in the rotating right-hand orthogonal coordinate system $Oxyz$ whose y-axis coincides with the radius vector constructed from the attracting center O to the point O_1, and whose x-axis coincides with the direction of motion in such a way that the vector of the absolute velocity of the interceptor's center of mass lies in the xy-plane. The orientation of the axes xyz relative to the inertial coordinates is defined (see Fig. 1) by the longitude Ω of the ascending node, the inclination i of the instantaneous orbital plane to the equator, and the range angle u. The equations of motion of the center of mass of the interceptor are

$$V_x^{\cdot} = W_x + \omega_z V_y, \qquad V_y^{\cdot} = W_y - \omega_z V_x - g \tag{1.1}$$
$$0 = W_z + \omega_y V_x, \quad \omega_z = -V_x / r, \; g = g_0 (R_0 / r)^2$$

The rates of change of the angles defining the orientation of the rotating axes relative to the inertial axes are given by the differential equations

$$\frac{d\Omega}{dt} = \omega_y \frac{\sin u}{\sin i}, \quad \frac{di}{dt} = \omega_y \cos u, \quad \frac{du}{dt} = -\omega_z - \omega_y \sin u \operatorname{ctg} i \tag{1.2}$$

We shall make our choice of the control law for the motion of the center of mass of the interceptor subject to the conditions of integrability of equations of motion (1.1), (1.2); moreover, we shall restrict its choice to the class of functions in which the control constants ensuring convergence of the spacecraft can be determined with sufficient